

ORBIFOLD QUANTUM D-MODULES ASSOCIATED TO WEIGHTED PROJECTIVE SPACES

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ABSTRACT. We construct in an abstract fashion the orbifold quantum cohomology (quantum orbifold cohomology) of weighted projective space, starting from the orbifold quantum differential operator. We obtain the product, grading, and intersection form by making use of the associated self-adjoint D-module and the Birkhoff factorization procedure. The method extends to the more difficult case of Fano hypersurfaces in weighted projective space. However, in contrast to the case of weighted projective space itself or a Fano hypersurface in projective space, a “small Birkhoff cell” can appear in the construction; we give an example of this phenomenon.

1. INTRODUCTION

The weighted projective space

$$\mathbb{P}(w_0, \dots, w_n) = \mathbb{C}^{n+1} - \{0\} / \mathbb{C}^*, \quad z \cdot (z_0, \dots, z_n) = (z^{-w_0} z_0, \dots, z^{-w_n} z_n)$$

provides a simple test case (see [3], [2], [12]) for the recently developed theories of orbifold cohomology and orbifold quantum cohomology. Direct geometrical calculations are difficult, but mirror symmetry suggests an alternative and very effective approach: Corti and Golyshev conjectured (see [6], [5]) that the structure constants can be read off from

$$T_w - q = \prod_{i=0}^n (w_i \hbar \partial) (w_i \hbar \partial - \hbar) \dots (w_i \hbar \partial - (w_i - 1) \hbar) - q,$$

where $\partial = q \frac{d}{dq}$; this is an ordinary differential operator of order $s = \sum_{i=0}^n w_i$.

This generalizes the well known quantum differential equation of projective space $\mathbb{C}P^n = \mathbb{P}(1, \dots, 1)$. Namely, the equation $((\hbar \partial)^{n+1} - q)y = 0$ is a scalar form of the system

$$\hbar \partial \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 0 & & & q \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix}$$

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where the matrix is interpreted as that of quantum multiplication by the generator $b \in H^2\mathbb{C}P^n$ with respect to the standard cohomology basis $1, b, \dots, b^n$. Thus,

$$b \circ b^i = \begin{cases} b^{i+1} & \text{if } 0 \leq i < n \\ q & \text{if } i = n \end{cases}$$

from which all quantum products $b^i \circ b^j$ can be computed.

The conjecture of Corti and Golyshev was proved in [3], by extending to orbifold quantum cohomology a well known method of Givental for quantum cohomology. The method has three steps. First, a basis of solutions of the quantum differential equation is written down — the I -function. Then, the orbifold version of Givental’s Mirror Theorem shows that the I -function is equal to the J -function, a certain generating function for Gromov-Witten invariants. This is the most substantial ingredient, but specific properties of weighted projective spaces are not required. Finally, the structure constants for the orbifold quantum product are extracted from this J -function by a method which involves repeated differentiation.

The first goal of this paper is to give a straightforward version (alluded to in the introduction to [3]) of the last step, using the Birkhoff factorization method of [8]. This amounts to using the differential equation (D-module) rather than its solution (I-function).

The second goal, and the main emphasis of the paper, is to study in its own right the differential equation $(T_w - q)y = 0$ (or rather, the D-module $D^h/(T_w - q)$, where D^h is a certain ring of differential operators). It is remarkable that such a simple differential operator contains all relevant geometrical information concerning the orbifold quantum cohomology, which is complicated and non-intuitive even in the case of $\mathbb{P}(w_0, \dots, w_n)$.

In section 2 we review some standard notation, and in section 3 we state the results of [3] for weighted projective spaces. In section 4, we give a systematic derivation of these results from the quantum differential equations.

In section 5 we indicate how the Birkhoff factorization method extends to hypersurfaces of Fano type in weighted projective spaces. This generalizes the method of [14] for Fano hypersurfaces in weighted projective spaces. It presents a new feature: instead of the “big cell” of the Birkhoff decomposition, in general a “small cell” is needed. Alternatively, this method can be interpreted as a “big cell factorization” followed by a Gram-Schmidt orthogonalization procedure.

The first author is very grateful to Alessio Corti for explaining the conjecture and the basic ideas of orbifold quantum cohomology; the

main idea for extracting the structure constants of $\mathbb{P}(w_0, \dots, w_n)$ was originally worked out with him, and Alessio also explained the geometry behind the hypersurface example for which a D-module interpretation is presented here in section 5. He also thanks Hiroshi Iritani for many essential explanations, and Josef Dorfmeister for discussions on the Birkhoff decomposition.

2. NOTATION FOR ORBIFOLD COHOMOLOGY

We write $\mathbb{P}(w_0, \dots, w_n) = \mathbb{P}(w)$ from now on. As far as possible we shall follow the notation of [3] for orbifold cohomology. First, let

$$\begin{aligned} F &= \left\{ \frac{i}{w_j} \mid 0 \leq i \leq w_j - 1, \quad 0 \leq j \leq n \right\} \\ &= \{f_1, \dots, f_k\} \text{ where } 0 = f_1 < f_2 < \dots < f_k < f_{k+1} \stackrel{\text{def}}{=} 1. \end{aligned}$$

Let u_1, \dots, u_k be the “multiplicities” of the fractions f_1, \dots, f_k as elements of F . We write

$$s = u_1 + \dots + u_k = w_0 + \dots + w_n.$$

The positive integer u_i can also be described as the cardinality of the set $S_{f_i} = \{j \mid w_j f_i \in \mathbb{Z}\} \subseteq \{0, \dots, n\}$.

The additive structure is given by

$$H_{\text{orbi}}^* \mathbb{P}(w) \cong \bigoplus_{i=1}^k H^* \mathbb{P}(V_{f_i})$$

where $V_{f_i} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid z_j = 0 \text{ if } j \notin S_{f_i}\} \cong \mathbb{C}^{u_i}$. The subspace of $H_{\text{orbi}}^* \mathbb{P}(w)$ corresponding to $H^* \mathbb{P}(V_{f_i})$ has a basis of the form

$$\mathbf{1}_{f_i}, \mathbf{1}_{f_i} p, \dots, \mathbf{1}_{f_i} p^{u_i-1},$$

where $p \in H_{\text{orbi}}^2 \mathbb{P}(w)$ and $\mathbf{1}_{f_i}$ denotes a certain class in $H_{\text{orbi}}^* \mathbb{P}(w)$. In particular, when $i = 1$ we have $u_1 = n+1$ and generators $\mathbf{1}_0, \mathbf{1}_0 p, \dots, \mathbf{1}_0 p^n$; we shall just write $1, p, \dots, p^n$ in this case.

The orbifold cohomology of $\mathbb{P}(w)$ is commutative and associative, with identity element 1. It has a natural grading, in which

$$|\mathbf{1}_{f_i} p^j| = |\mathbf{1}_{f_i}| + |p^j| = 2 \text{ age } \mathbf{1}_{f_i} + 2j.$$

Here, $\text{age } \mathbf{1}_{f_i} = (u_1 + \dots + u_{i-1}) - f_i s = \langle -w_0 f_i \rangle + \dots + \langle -w_n f_i \rangle$ where $\langle r \rangle = r - \max\{i \in \mathbb{Z} \mid i \leq r\}$. The orbifold cohomology also has a nondegenerate symmetric “intersection pairing” $(\ , \)$, which generalizes the Poincaré pairing for ordinary cohomology.

We record the following properties for later use.

Lemma 2.1.

(1) $f_i + f_j = 1$ if $i + j = k + 2$.

(2) $u_i = u_j$ if $i + j = k + 2$.

(3) $u_2 + \cdots + u_i = u_{k+2-i} + \cdots + u_k$ for $2 \leq i \leq k$.

Proof. The involution $f \mapsto 1 - f$ preserves $F \cup \{1\}$. It maps $f_1 < \cdots < f_{k+1}$ to $1 - f_{k+1} < \cdots < 1 - f_1$, so these sequences must coincide. This proves (1), then (2) and (3) follow immediately. \square

3. THE STRUCTURE CONSTANTS: STATEMENT OF RESULTS

As mentioned in the introduction, a key role is played by the s -th order differential operator

$$\begin{aligned} T_w - q &= \prod_{i=0}^n (w_i \hbar \partial) (w_i \hbar \partial - \hbar) \cdots (w_i \hbar \partial - (w_i - 1)\hbar) - q \\ &= w^w \hbar^s \prod_{i=0}^n \partial \left(\partial - \frac{1}{w_i} \right) \cdots \left(\partial - \frac{w_i - 1}{w_i} \right) - q, \end{aligned}$$

where $s = \sum_{i=0}^n w_i$, $w^w = \prod_{i=0}^n w_i^{w_i}$, and $\partial = q \frac{d}{dq}$. This notation for the parameter \hbar and the complex variable q is standard.

In this section we state without explanation how the structure constants of orbifold quantum cohomology — and, in particular, of orbifold cohomology itself — may be extracted from the differential operator $T_w - q$. A systematic explanation will be given in the next section.

Using the formula $\partial q^{-1} = q^{-1}(\partial - 1)$, we may factorize the differential operator $q^{-1}T_w$ in the following way:

$$q^{-1}T_w = \underbrace{m_k q^{-\Delta_k} (\hbar \partial)^{u_k}}_{k\text{th factor}} \underbrace{m_{k-1} q^{-\Delta_{k-1}} (\hbar \partial)^{u_{k-1}}}_{k-1\text{th factor}} \cdots \underbrace{m_1 q^{-\Delta_1} (\hbar \partial)^{u_1}}_{1\text{st factor}}$$

where

$$\Delta_i = f_{i+1} - f_i, \quad m_i = \prod_{j \in S_{f_i}} w_j,$$

for $1 \leq i \leq k$. Thus we have $\prod_{i=1}^k m_i = w^w$ and $\sum_{i=1}^k \Delta_i = 1$. We shall need the following symmetry properties later on, which follow directly from Lemma 2.1:

Lemma 3.1.

(1) $\Delta_i = \Delta_j$ if $i + j = k + 1$.

(2) $m_i = m_j$ if $i + j = k + 2$.

Let us rewrite the factorization above as

$$q^{-1}T_w = \frac{1}{r_s} \hbar \partial \frac{1}{r_{s-1}} \hbar \partial \cdots \frac{1}{r_1} \hbar \partial$$

where:

Definition 3.2. For $1 \leq \alpha \leq s$,

$$r_\alpha = \begin{cases} \frac{1}{m_i} q^{\Delta_i} & \text{if } \alpha = u_1 + \cdots + u_i \\ 1 & \text{otherwise.} \end{cases}$$

Then the result of [3] may be stated as follows:

Theorem 3.3. Denote by c_0, \dots, c_{s-1} the additive basis

$$1, p, \dots, p^{u_1-1}; \mathbf{1}_{f_2}, \mathbf{1}_{f_2} p, \dots, \mathbf{1}_{f_2} p^{u_2-1}; \dots; \mathbf{1}_{f_k}, \mathbf{1}_{f_k} p, \dots, \mathbf{1}_{f_k} p^{u_k-1}$$

of $H_{\text{orbi}}^* \mathbb{P}(w)$. Then the matrix of orbifold quantum multiplication by p with respect to this basis is given by

$$\begin{pmatrix} 0 & & & r_s \\ r_1 & \ddots & & \\ & \ddots & \ddots & \\ & & r_{s-1} & 0 \end{pmatrix}.$$

That is, we have $p \circ c_i = r_{i+1} c_{i+1}$ for $0 \leq i < s-1$ and $p \circ c_{s-1} = r_s c_0$. In particular, p is a cyclic generator of this ring.

The orbifold structure constants (giving the product structure of $H_{\text{orbi}}^* \mathbb{P}(w)$) are obtained by setting $q = 0$ in the above matrix. Although the matrix itself gives only the products involving p , all other products can be deduced.

4. DIRECT APPROACH FROM THE D-MODULE

The structure constants in Theorem 3.3 were computed in [3] from the I-function (i.e. solution of the differential equation $(T_w - q)y = 0$) and the mirror theorem of Givental. In this section we discuss a somewhat different procedure: we construct “abstract orbifold quantum cohomology” from $T_w - q$ itself. To prove that our abstract orbifold quantum cohomology agrees with the usual orbifold quantum cohomology, it is still necessary to appeal to the mirror theorem, so in this sense our procedure relates only to the extraction of information from the differential equation. However, our procedure is probably the most direct way, especially in the case of hypersurfaces, of obtaining the orbifold degrees and orbifold Poincaré pairing as well as the structure constants.

We follow [8] and chapter 6 of [9], although the orbifold case presents some new features. Let us consider the the D^h -module

$$\mathcal{M} = D^h / (T_w - q)$$

where D^h denotes the ring of (ordinary) differential operators generated by $\hbar \partial$ with coefficients in the ring \mathcal{O} of functions which are meromorphic in a neighbourhood of $q = 0$ and holomorphic in a neighbourhood of $\hbar = 0$. Here, $(T_w - q)$ denotes the left ideal generated by $T_w - q$.

The D^{\hbar} -module \mathcal{M} is free of rank s over \mathcal{O} . With respect to the natural basis $1, \hbar\partial, \dots, (\hbar\partial)^{s-1}$, the matrix of the action of ∂ is of the form

$$\Omega = \frac{1}{\hbar}\omega + \theta^{(0)} + \hbar\theta^{(1)} + \dots.$$

More precisely, if we identify \mathcal{M} with the space of meromorphic sections of the trivial bundle, we may regard $1, \hbar\partial, \dots, (\hbar\partial)^{s-1}$ as a local basis of sections, and the action of ∂ on \mathcal{M} defines a connection on the bundle, with local connection matrix Ω .

If we replace $\hbar\partial$ by an abstract (commutative) variable p , then set $\hbar = 0$, we obtain from \mathcal{M} a commutative ring generated by p which is subject to the relation $w^w p^s - q$, and with a distinguished basis $1, p, \dots, p^{s-1}$. Restricting the coefficients to be polynomials in $q^{1/s}$ for convenience, we obtain the “abstract orbifold quantum cohomology ring”

$$QA = \mathbb{C}[p, q^{1/s}]/(w^w p^s - q).$$

The matrix of Theorem 3.3 does indeed satisfy the relation $w^w p^s - q$, so our ring QA is isomorphic to the orbifold quantum cohomology ring of $\mathbb{P}(w)$.

In order to define “abstract orbifold Gromov-Witten invariants” (structure constants) we shall introduce a ring A , the “abstract orbifold cohomology ring”, such that $QA \cong A \otimes \mathbb{C}[q^{1/s}]$. A choice of basis of each ring will give a specific isomorphism $\delta : QA \cong A \otimes \mathbb{C}[q^{1/s}]$, hence an $A \otimes \mathbb{C}[q^{1/s}]$ -valued product operation

$$a \circ b = \delta(\delta^{-1}(a)\delta^{-1}(b))$$

on A with all the expected properties of the orbifold quantum product.

For this, our main task will be the construction of a suitable basis. The main step is to transform the basis $1, \hbar\partial, \dots, (\hbar\partial)^{s-1}$ to a new basis, with respect to which the connection matrix has the form

$$\hat{\Omega} = \frac{1}{\hbar}\hat{\omega}$$

where $\hat{\omega}$ is independent of \hbar . In the case of a Fano manifold, the transformation procedure is explained in detail in chapter 6 of [9]. It involves a Birkhoff factorization $L = L_- L_+$ of a matrix-valued function L such that $\Omega = L^{-1}dL$, after which one defines $\hat{\Omega} = (L_-)^{-1}dL_-$. The basis $1, \hbar\partial, \dots, (\hbar\partial)^{s-1}$ is transformed to the new basis $L_+^{-1} \cdot 1, L_+^{-1} \cdot \hbar\partial, \dots, L_+^{-1} \cdot (\hbar\partial)^{s-1}$, where $L_+^{-1} \cdot (\hbar\partial)^i$ means $\sum_{j=0}^{s-1} (L_+)^{-1}_{ji} (\hbar\partial)^j$.

In general it is difficult to carry out such Birkhoff factorizations. However, the factorizations needed in the present article can be carried out explicitly by the finite algorithm given in [1] and section 6.6 of [9]. In the case of weighted projective spaces themselves (though not for hypersurfaces), the differential operator factorization given in section

3 provides a short cut to an immediate explicit answer. Namely, we introduce directly a new basis P_0, \dots, P_{s-1} by defining

$$P_0 = 1 \text{ and } P_i = \frac{1}{r_i} \hbar \partial P_{i-1}$$

for $1 \leq i \leq s-1$. By construction, with respect to this basis of $D^h/(T_w)$, the matrix of ∂ has the form $\frac{1}{\hbar} \hat{\omega}$, where $\hat{\omega}$ is the matrix of Theorem 3.3.

The Birkhoff factorization is given implicitly by this. Namely, L_+ may be read off by regarding the above basis as $L_+^{-1} \cdot 1, L_+^{-1} \cdot \hbar \partial, \dots, L_+^{-1} \cdot (\hbar \partial)^{d-1}$. We have $L_+ = Q_0(I + \hbar Q_1 + \dots + \hbar^{k-2} Q_{k-2})$ where

$$Q_0 = \begin{pmatrix} \frac{1}{m_0} q^{f_1} I & & & \\ & \frac{1}{m_0 m_1} q^{f_2} I & & \\ & & \ddots & \\ & & & \frac{1}{m_0 \dots m_{k-1}} q^{f_k} I \end{pmatrix},$$

where $m_0 = 1$ and Q_1, \dots, Q_{k-2} are constant.

We shall use the above basis to construct in turn a product operation, a grading, and a pairing.

1. The product

Let us group the basis elements of \mathcal{M} as follows:

$$\begin{aligned} & (\hbar \partial)^i \quad \text{for } 0 \leq i \leq u_1 - 1 \\ & (\hbar \partial)^i m_1 q^{-\Delta_1} (\hbar \partial)^{u_1} \quad \text{for } 0 \leq i \leq u_2 - 1 \\ & \dots \\ & (\hbar \partial)^i m_1 q^{-\Delta_1} (\hbar \partial)^{u_1} \dots m_{k-1} q^{-\Delta_{k-1}} (\hbar \partial)^{u_{k-1}-1} \quad \text{for } 0 \leq i \leq u_k - 1 \end{aligned}$$

Replacing $\hbar \partial$ by p we obtain a corresponding basis of QA . We introduce the algebra A by declaring that this basis of QA corresponds to the following basis of $A \otimes \mathbb{C}[q^{1/s}]$:

$$\begin{aligned} & 1, p, \dots, p^{u_1-1}; \\ & \mathbf{1}_{f_2}, \mathbf{1}_{f_2} p, \dots, \mathbf{1}_{f_2} p^{u_2-1}; \\ & \dots \\ & \mathbf{1}_{f_k}, \mathbf{1}_{f_k} p, \dots, \mathbf{1}_{f_k} p^{u_k-1} \end{aligned}$$

By definition, the action of p on $A \otimes \mathbb{C}[q^{1/s}]$ is given (with respect to this basis) by the matrix of Theorem 3.3. As 1 is a cyclic generator, this action extends to a product operation on $A \otimes \mathbb{C}[q^{1/s}]$, that is, it allows us to define the product of any two elements $\mathbf{1}_{f_i} p^j, \mathbf{1}_{f_k} p^l$. We denote this product by $\mathbf{1}_{f_i} p^j \circ \mathbf{1}_{f_k} p^l$, and regard $A \otimes \mathbb{C}[q^{1/s}]$ as the abstract

orbifold quantum cohomology ring of $\mathbb{P}(w)$. A concrete example is given below.

2. The grading

The differential operator $T_w - q$ is homogeneous of weight $2s$, if we assign weights as follows: $|\hbar| = 2$, $|\partial| = 0$, $|q| = 2s$. The differential operators P_0, \dots, P_{s-1} are also homogeneous. Indeed, from the formula for $P_{u_1+\dots+u_i}$, its weight is

$$\begin{aligned} |P_{u_1+\dots+u_i}| &= 2(u_1 + \dots + u_i) - 2s(\Delta_1 + \dots + \Delta_i) \\ &= 2(u_1 + \dots + u_i) - 2sf_{i+1} \\ &= 2 \operatorname{age} \mathbf{1}_{f_{i+1}}. \end{aligned}$$

It follows that our product operation satisfies

$$|\mathbf{1}_{f_i} p^j \circ \mathbf{1}_{f_k} p^l| = |\mathbf{1}_{f_i} p^j| + |\mathbf{1}_{f_k} p^l|$$

where $|\cdot|$ denotes the usual orbifold quantum cohomology grading.

3. Self-adjointness and the pairing

We shall obtain a natural identification of the D^\hbar -module $\mathcal{M} = D^\hbar/(T_w - q)$ with a “dual” D^\hbar -module; this will give us a pairing on \mathcal{M} , and a nondegenerate symmetric bilinear form on A . This pairing will turn out to be a $\mathbb{C}[q^{1/s}]$ -linear extension of a pairing on A . We shall use the notation of section 6.3 of [9].

First, the D^\hbar -module \mathcal{M}^* is defined to be the space of \mathcal{O} -module homomorphisms $\mathcal{M} \rightarrow \mathcal{O}$. The D^\hbar -module structure is given by

$$(\hbar \cdot \pi)(P) = \hbar \pi(P), \quad (\partial_i \cdot \pi)(P) = -\pi(\partial_i \cdot P) + \frac{\partial}{\partial t_i} \pi(P)$$

for $\pi \in \mathcal{M}^*$.

Next, we denote by $\bar{\mathcal{M}}^*$ the D^\hbar -module obtained from \mathcal{M}^* by reversing the sign in the action of \hbar . That is, $\bar{\mathcal{M}}^* = \mathcal{M}^*$ (as \mathcal{O} -modules), but with action of D^\hbar given by $\hbar \odot \pi = -\hbar \pi$, $\partial_i \odot \pi = \partial_i \cdot \pi$.

Let $\delta_0, \dots, \delta_{s-1}$ be the basis of $\bar{\mathcal{M}}^* = \mathcal{M}^*$ (over \mathcal{O}) which is dual to the basis $1, \hbar \partial, \dots, (\hbar \partial)^{s-1}$ of \mathcal{M} .

Proposition 4.1.

- (1) δ_n is a cyclic element of $\bar{\mathcal{M}}^*$ (that is, $D^\hbar \odot \delta_n = \bar{\mathcal{M}}^*$).
- (2) $(T_w - q) \odot \delta_n = 0$.
- (3) The map $\mathcal{M} \rightarrow \bar{\mathcal{M}}^*$, $[P] \mapsto [P \odot \delta_n]$ is an isomorphism of D^\hbar -modules.

Proof. Let P_0^*, \dots, P_{s-1}^* be the basis of $\bar{\mathcal{M}}^*$ which is dual to P_0, \dots, P_{s-1} . For readability we shall omit square brackets throughout this proof. Note that $P_i^* = \delta_i$ for $i = 0, \dots, n$.

We claim that

$$P_\alpha \odot \delta_n = \begin{cases} P_{n-\alpha}^* = \delta_{n-\alpha} & \text{when } 0 \leq \alpha < u_1 = n+1, \\ \frac{m_1}{m_{i+1}} P_{s+n-\alpha}^* & \text{when } u_1 + \cdots + u_i \leq \alpha < u_1 + \cdots + u_{i+1}, \\ P_n^* = \delta_n & \text{when } \alpha = s. \end{cases}$$

Assuming this, the first two formulae (for $\alpha = 0, \dots, s-1$) prove (1). In the third formula $P_n^* = \delta_n$, P_s means $\frac{1}{r_s} \hbar \partial \frac{1}{r_{s-1}} \hbar \partial \dots \frac{1}{r_1} \hbar \partial$, which is $q^{-1} T_w$, so this gives (2). The third statement is an immediate consequence of (1) and (2) (cf. section 6.3 of [9]).

To prove the claim, we shall make use of

$$\hbar \partial P_\alpha = r_{\alpha+1} P_{\alpha+1} \quad (*)$$

$$\hbar \partial \odot P_\alpha^* = r_\alpha P_{\alpha-1}^* \quad (**)$$

and the value of r_α given in Definition 3.2.

The case $0 \leq \alpha < u_1 = n+1$.

Since $r_0 = \cdots = r_n = 1$, from (**) we have $P_\alpha \odot \delta_n = (\hbar \partial)^\alpha \odot P_n^* = P_{n-\alpha}^*$.

The case $u_1 + \cdots + u_i \leq \alpha < u_1 + \cdots + u_{i+1}$.

We shall prove this by induction on $i = 0, 1, \dots, k-1$ (regarding the previous case as $i = 0$).

(i) If $\alpha = u_1 + \cdots + u_i$ for some $i \geq 1$, we have

$$\begin{aligned} P_\alpha \odot \delta_n &= m_i q^{-\Delta_i} \hbar \partial P_{\alpha-1} \odot \delta_n \quad \text{by } (*), \text{ as } r_\alpha = m_i^{-1} q^{\Delta_i} \\ &= m_i q^{-\Delta_i} \hbar \partial \odot \frac{m_1}{m_i} P_{s+n-(\alpha-1)}^* \quad (\text{inductive hypothesis}) \\ &= m_1 q^{-\Delta_i} r_{s+n-\alpha+1} P_{s+n-\alpha}^* \quad \text{by } (**). \end{aligned}$$

Now, $s+n-\alpha+1 = s+u_1 - (u_1 + \cdots + u_i) = s - (u_{k+2-i} + \cdots + u_k)$ (by Lemma 2.1) $= u_1 + \cdots + u_{k+1-i}$. (This argument applies only if $i \geq 2$, but the case $i = 1$ is obvious.) Hence

$$r_{s+n-\alpha+1} = r_{u_1 + \cdots + u_{k+1-i}} = \frac{1}{m_{k+1-i}} q^{\Delta_{k+1-i}} = \frac{1}{m_{i+1}} q^{\Delta_i}$$

by Lemma 3.1. We obtain $P_\alpha \odot \delta_n = \frac{m_1}{m_{i+1}} P_{s+n-\alpha}^*$.

(ii) If $u_1 + \cdots + u_i \leq \alpha < u_1 + \cdots + u_{i+1}$ for some i , then

$$\begin{aligned} P_\alpha \odot \delta_n &= \hbar \partial P_{\alpha-1} \odot \delta_n \quad \text{by } (*), \text{ as } r_\alpha = 1 \\ &= \hbar \partial \odot \frac{m_1}{m_{i+1}} P_{s+n-(\alpha-1)}^* \quad (\text{inductive hypothesis}) \\ &= \frac{m_1}{m_{i+1}} r_{s+n-\alpha+1} P_{s+n-\alpha}^* \quad \text{by } (**). \end{aligned}$$

Here we have $s+n-\alpha+1 = u_1 + \cdots + u_{k+1-i} - l$ with $0 < l < u_{i+1} = u_{k+1-i}$ (from Lemma 2.1), so $r_{s+n-\alpha+1} = 1$. We obtain $P_\alpha \odot \delta_n = \frac{m_1}{m_{i+1}} P_{s+n-\alpha}^*$ again.

The case $\alpha = s$.

We have

$$\begin{aligned}
P_s \odot \delta_n &= m_k q^{-\Delta_k} \hbar \partial P_{s-1} \odot \delta_n \quad \text{by } (*) \\
&= m_k q^{-\Delta_k} \hbar \partial \odot \frac{m_1}{m_k} P_{n+1}^* \quad (\text{inductive hypothesis}) \\
&= m_1 q^{-\Delta_k} r_{n+1} P_n^* \quad \text{by } (**).
\end{aligned}$$

Here we have $r_{n+1} = r_{u_1} = \frac{1}{m_1} q^{\Delta_1}$, and $\Delta_1 = \Delta_k$ by Lemma 3.1, so we conclude that $P_s \odot \delta_n = \delta_n$. \square

The natural composition $\mathcal{M} \times \mathcal{M} \rightarrow \bar{\mathcal{M}}^* \times \mathcal{M} \rightarrow \mathcal{O}$, making use of the above isomorphism $\mathcal{M} \rightarrow \bar{\mathcal{M}}^*$, defines a pairing. We normalize it as follows:

Definition 4.2. $\langle\langle P, Q \rangle\rangle = \frac{1}{w_0 \dots w_n} (P \odot \delta_n)(Q) \quad (= \frac{1}{m_1} (P \odot \delta_n)(Q)).$

Corollary 4.3. *We have*

$$\langle\langle P_\alpha, P_\beta \rangle\rangle = \begin{cases} \frac{1}{m_1} \delta_{n-\alpha, \beta} & \text{if } 0 \leq \alpha < u_1, \\ \frac{1}{m_{i+1}} \delta_{s+n-\alpha, \beta} & \text{if } u_1 + \dots + u_i \leq \alpha < u_1 + \dots + u_{i+1}, i \geq 1. \end{cases}$$

Our normalization ensures that the induced pairing on A agrees with the usual Poincaré intersection pairing on the cohomology of $\mathbb{P}(w)$; it is known from [11] that $(1, p^n) = 1/(w_0 \dots w_n)$. The induced pairing on $A \otimes \mathbb{C}[q^{1/s}]$ satisfies the Frobenius property (see section 6.5 of [9]). Hence, by the cyclic property, it agrees with the orbifold quantum Poincaré intersection pairing.

This concludes our construction of an abstract orbifold quantum product, grading, and pairing directly from $T_w - q$, and our verification that they agree with the usual ones.

Example 4.4. $\mathbb{P}(1, 2, 3)$

We have $w_0 = 1, w_1 = 2, w_2 = 3$ and $s = 1 + 2 + 3 = 6$. The differential operator is

$$\begin{aligned}
T_w - q &= \hbar \partial \ 2\hbar \partial (2\hbar \partial - \hbar) \ 3\hbar \partial (3\hbar \partial - \hbar) (3\hbar \partial - 2\hbar) - q \\
&= 2^2 3^3 \hbar^6 \partial^3 (\partial - \frac{1}{3}) (\partial - \frac{1}{2}) (\partial - \frac{2}{3}) - q.
\end{aligned}$$

This has order 6, and it is homogeneous of weight 12, where $|\hbar| = 2, |q| = 12$.

We have $F = \{\frac{0}{1}, \frac{0}{2}, \frac{1}{2}, \frac{0}{3}, \frac{1}{3}, \frac{2}{3}\} = \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$, so $u_1 = 3, u_2 = 1, u_3 = 1, u_4 = 1$. It is convenient to display all relevant data in the following

diagram:

	$w_0 = 1$	$w_1 = 2$	$w_2 = 3$	
$S_{f_1} = \{0, 1, 2\}, f_1 = 0$	$\frac{0}{1}$	$\frac{0}{2}$	$\frac{0}{3}$	$\Delta_1 = \frac{1}{3}, m_1 = 6$
$S_{f_2} = \{2\}, f_2 = \frac{1}{3}$			$\frac{1}{3}$	$\Delta_2 = \frac{1}{6}, m_2 = 3$
$S_{f_3} = \{1\}, f_3 = \frac{1}{2}$		$\frac{1}{2}$		$\Delta_3 = \frac{1}{6}, m_3 = 2$
$S_{f_4} = \{2\}, f_4 = \frac{2}{3}$			$\frac{2}{3}$	$\Delta_4 = \frac{1}{3}, m_4 = 3$

In the central 4×3 block, the number of entries in the i -th row is u_i , and the number of entries in the j -th column is w_{j-1} .

The factorization is

$$q^{-1}T_w = 3q^{-\frac{1}{3}}(\hbar\partial)^1 2q^{-\frac{1}{6}}(\hbar\partial)^1 3q^{-\frac{1}{6}}(\hbar\partial)^1 6q^{-\frac{1}{3}}(\hbar\partial)^3.$$

The bases of \mathcal{M} and A constructed above are:

$$\begin{array}{ll} 1, \hbar\partial, (\hbar\partial)^2 & 1, p, p^2 \\ 6q^{-\frac{1}{3}}(\hbar\partial)^3 & \mathbf{1}_{\frac{1}{3}} \\ 3q^{-\frac{1}{6}}(\hbar\partial) \ 6q^{-\frac{1}{3}}(\hbar\partial)^3 & \mathbf{1}_{\frac{1}{2}} \\ 2q^{-\frac{1}{6}}\hbar\partial \ 3q^{-\frac{1}{6}}\hbar\partial \ 6q^{-\frac{1}{3}}(\hbar\partial)^3 & \mathbf{1}_{\frac{2}{3}} \end{array}$$

The matrix of structure constants (quantum multiplication by p) with respect to this basis is

$$\begin{pmatrix} 0 & & & & & \frac{1}{3}q^{\frac{1}{3}} \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & \frac{1}{6}q^{\frac{1}{3}} & 0 & & \\ & & & \frac{1}{3}q^{\frac{1}{6}} & 0 & \\ & & & & \frac{1}{2}q^{\frac{1}{6}} & 0 \end{pmatrix}.$$

These products determine all others, and we obtain the following orbifold quantum multiplication table:

	1	p	p^2	$\mathbf{1}_{\frac{1}{3}}$	$\mathbf{1}_{\frac{1}{2}}$	$\mathbf{1}_{\frac{2}{3}}$
1	1	p	p^2	$\mathbf{1}_{\frac{1}{3}}$	$\mathbf{1}_{\frac{1}{2}}$	$\mathbf{1}_{\frac{2}{3}}$
p		p^2	$\frac{1}{6}q^{\frac{1}{3}}\mathbf{1}_{\frac{1}{3}}$	$\frac{1}{3}q^{\frac{1}{6}}\mathbf{1}_{\frac{1}{2}}$	$\frac{1}{2}q^{\frac{1}{6}}\mathbf{1}_{\frac{2}{3}}$	$\frac{1}{3}q^{\frac{1}{3}}$
p^2			$\frac{1}{18}q^{\frac{1}{2}}\mathbf{1}_{\frac{1}{2}}$	$\frac{1}{6}q^{\frac{1}{3}}\mathbf{1}_{\frac{2}{3}}$	$\frac{1}{6}q^{\frac{1}{2}}$	$\frac{1}{3}q^{\frac{1}{3}}p$
$\mathbf{1}_{\frac{1}{3}}$				$\frac{1}{3}q^{\frac{1}{3}}$	$q^{\frac{1}{6}}p$	$2p^2$
$\mathbf{1}_{\frac{1}{2}}$					$3p^2$	$q^{\frac{1}{6}}\mathbf{1}_{\frac{1}{3}}$
$\mathbf{1}_{\frac{2}{3}}$						$\frac{2}{3}q^{\frac{1}{6}}\mathbf{1}_{\frac{1}{2}}$

Orbifold cohomology products are obtained by setting $q = 0$ in this table. Note that p generates the orbifold quantum cohomology, but not the orbifold cohomology. Ages and degrees are as shown below:

age $\mathbf{1}_0 = 0$	$ 1 = 0$	$ p = 2$	$ p^2 = 4$
age $\mathbf{1}_{\frac{1}{3}} = 1$			$ \mathbf{1}_{\frac{1}{3}} = 2$
age $\mathbf{1}_{\frac{1}{2}} = 1$		$ \mathbf{1}_{\frac{1}{2}} = 2$	
age $\mathbf{1}_{\frac{2}{3}} = 1$			$ \mathbf{1}_{\frac{2}{3}} = 2$

Finally, the pairing on \mathcal{M} is given by $\langle\langle P_i, P_j \rangle\rangle = \frac{1}{6}$ if $i + j = 2$, $\langle\langle P_3, P_5 \rangle\rangle = \langle\langle P_5, P_3 \rangle\rangle = \frac{1}{3}$, and $\langle\langle P_4, P_4 \rangle\rangle = \frac{1}{2}$ (with all other products zero). \square

Example 4.5. $\mathbb{P}(1, 1, 3)$

In this case we have orbifold classes with fractional degrees. We just state the results, as the calculations are very similar to those in the previous example. First, the data is

	$w_0 = 1$	$w_1 = 1$	$w_2 = 3$	
$S_{f_1} = \{0, 1, 2\}, f_1 = 0$	$\frac{0}{1}$	$\frac{0}{1}$	$\frac{0}{3}$	$\Delta_1 = \frac{1}{3}, m_1 = 3$
$S_{f_2} = \{2\}, f_2 = \frac{1}{3}$			$\frac{1}{3}$	$\Delta_2 = \frac{1}{3}, m_2 = 3$
$S_{f_3} = \{2\}, f_3 = \frac{2}{3}$			$\frac{2}{3}$	$\Delta_3 = \frac{1}{3}, m_3 = 3$

and we have

$$q^{-1}T_w = q^{-1}3^3\hbar^5\partial^3(\partial - \frac{1}{3})(\partial - \frac{2}{3}) = 3q^{-\frac{1}{3}}(\hbar\partial)^1 3q^{-\frac{1}{3}}(\hbar\partial)^1 3q^{-\frac{1}{3}}(\hbar\partial)^3.$$

The orbifold quantum multiplication table is

	1	p	p^2	$\mathbf{1}_{\frac{1}{3}}$	$\mathbf{1}_{\frac{2}{3}}$
1	1	p	p^2	$\mathbf{1}_{\frac{1}{3}}$	$\mathbf{1}_{\frac{2}{3}}$
p		p^2	$\frac{1}{3}q^{\frac{1}{3}}\mathbf{1}_{\frac{1}{3}}$	$\frac{1}{3}q^{\frac{1}{3}}\mathbf{1}_{\frac{2}{3}}$	$\frac{1}{3}q^{\frac{1}{3}}$
p^2			$\frac{1}{9}q^{\frac{2}{3}}\mathbf{1}_{\frac{2}{3}}$	$\frac{1}{9}q^{\frac{2}{3}}$	$\frac{1}{3}q^{\frac{1}{3}}p$
$\mathbf{1}_{\frac{1}{3}}$				$\frac{1}{3}q^{\frac{1}{3}}p$	p^2
$\mathbf{1}_{\frac{2}{3}}$					$\mathbf{1}_{\frac{1}{3}}$

where $1, p, p^2, \mathbf{1}_{\frac{1}{3}}, \mathbf{1}_{\frac{2}{3}}$ correspond to $1, \hbar\partial, (\hbar\partial)^2, 3q^{-\frac{1}{3}}(\hbar\partial)^3, 3q^{-\frac{1}{3}}(\hbar\partial)3q^{-\frac{1}{3}}(\hbar\partial)^3$.

We have

age $\mathbf{1}_0 = 0$	$ 1 = 0$	$ p = 2$	$ p^2 = 4$
age $\mathbf{1}_{\frac{1}{3}} = \frac{4}{3}$			$ \mathbf{1}_{\frac{1}{3}} = \frac{8}{3}$
age $\mathbf{1}_{\frac{2}{3}} = \frac{2}{3}$			$ \mathbf{1}_{\frac{2}{3}} = \frac{4}{3}$

and the pairing is given by $\langle\langle P_i, P_j \rangle\rangle = \frac{1}{3}$ if $i + j = 2$, $\langle\langle P_3, P_4 \rangle\rangle = \langle\langle P_4, P_3 \rangle\rangle = \frac{1}{3}$ (with all other products zero). \square

5. HYPERSURFACES IN WEIGHTED PROJECTIVE SPACE

Corti and Golyshev conjectured that the orbifold quantum cohomology of a hypersurface¹

$$X^d \subseteq \mathbb{P}(w)$$

of degree d is governed by the differential operator

$$\hbar^s \prod_{i=0}^n (w_i \partial) (w_i \partial - 1) \dots (w_i \partial - (w_i - 1)) - q \hbar^d (d \partial + 1) \dots (d \partial + (d - 1)) (d \partial + d).$$

(this operator appears in section 7.3 of [6] without the \hbar factors; also in [5] for the Calabi-Yau case $s = d$, where the \hbar factors cancel out). The method of [3] (based on Givental's Mirror Theorem) supports this conjecture in the Fano case, i.e. when $s > d$.

In the spirit of the preceding sections, we shall give a method which extracts a canonical “abstract orbifold quantum cohomology” ring from

¹It is assumed that the hypersurface is positioned so that it inherits all singularities of $\mathbb{P}(w)$.

this differential operator. We verify that it gives the orbifold quantum cohomology of $X^3 \subseteq \mathbb{P}(1, 1, 1, 2)$, a nontrivial example which has been computed geometrically by Corti ([4]). We shall always assume that $s > d$, although our approach applies also when $s = d$ (cf. section 6.7 of [9]).

Since $\partial q = q(\partial + 1)$, we have

$$q\hbar^d d^d (\partial + \frac{1}{d}) \dots (\partial + \frac{d-1}{d}) (\partial + \frac{d}{d}) = \hbar^d d^d (\partial - \frac{d-1}{d}) \dots (\partial - \frac{1}{d}) (\partial - \frac{0}{d}) q,$$

which shows that both summands of

$$w^w \hbar^s \prod_{i=0}^n \partial (\partial - \frac{1}{w_i}) \dots (\partial - \frac{w_i-1}{w_i}) - q\hbar^d d^d (\partial + \frac{1}{d}) \dots (\partial + \frac{d-1}{d}) (\partial + \frac{d}{d})$$

contain a factor of $\hbar\partial$. Cancelling this factor from the left hand side, we obtain an operator of order $s - 1$ (in terms of D^\hbar -modules, we quotient out by the trivial D^\hbar -module $D^\hbar/(\hbar\partial)$). We call² this operator $T_{w_1, \dots, w_n} - qS_{d-1}$:

$$\underbrace{w^w \hbar^{s-1} \prod_{i=1}^n \partial (\partial - \frac{1}{w_i}) \dots (\partial - \frac{w_i-1}{w_i})}_{T_{w_1, \dots, w_n}} - \underbrace{q \hbar^{d-1} d^d (\partial + \frac{1}{d}) \dots (\partial + \frac{d-1}{d})}_{S_{d-1}}$$

Here we have assumed that $w_0 = 1$. *For simplicity, we shall also assume that w_1, \dots, w_n are such that no further left-cancellations of the above type are possible.* It follows that the D^\hbar -module

$$\mathcal{M} = D^\hbar / (T_{w_1, \dots, w_n} - qS_{d-1})$$

is irreducible. In the general case, an irreducible D^\hbar -module is obtained by left-cancelling all common factors (see [6]), and our method can in principle be applied to that.

Observe that the case $d = 1$ gives $T_{w_1, \dots, w_n} - q$, which is the operator associated with $\mathbb{P}(w_1, \dots, w_n)$, as expected. The case $w_1 = \dots = w_n = 1$ (hence $s = n + 1$) gives $(\hbar\partial)^n - qS_{d-1}$, which is the operator associated with a degree d hypersurface in $\mathbb{C}P^n$, denoted by M_{n+1}^d in [14].

As in section 2, we define

$$\begin{aligned} F &= \{ \frac{i}{w_j} \mid 0 \leq i \leq w_j - 1, \ 1 \leq j \leq n \} \\ &= \{ f_1, \dots, f_k \}. \end{aligned}$$

and denote by u_1, \dots, u_k the multiplicities of f_1, \dots, f_k . However, $u_1 = n$ here. We use the notation Δ_i, m_i as in section 3. Thus, we have a factorization

$$q^{-1} T_{w_1, \dots, w_n} = \frac{1}{r_{s-1}} \hbar\partial \frac{1}{r_{s-2}} \hbar\partial \dots \frac{1}{r_1} \hbar\partial$$

²The abbreviation T_w always means T_{w_0, \dots, w_n} .

and we can introduce $P_0 = 1$ and $P_i = \frac{1}{r_i} \hbar \partial P_{i-1}$ for $1 \leq i \leq s-2$. The equivalence classes of the operators P_0, \dots, P_{s-2} form a basis of the D^\hbar -module $D^\hbar / (T_{w_1, \dots, w_n} - qS_{d-1})$.

As in section 4, the action of ∂ defines a connection on the bundle whose space of sections is \mathcal{M} . However, when $d \geq 2$, the connection matrix Ω with respect to the basis P_0, \dots, P_{s-2} is not of the form $\frac{1}{\hbar} \hat{\omega}$. To achieve this form (which is the starting point for the construction of a product operation) we must construct a new basis.

It will be convenient to construct such a basis in two steps.

Step 1 The method of [14] produces a basis $\hat{P}_0, \dots, \hat{P}_{s-2}$, with respect to which the connection matrix has the form $\frac{1}{\hbar} \hat{\omega}$. Let us review that method here. As in our discussion of the Birkhoff factorization method in section 4, the new basis is given by $L_+^{-1} \cdot P_0, L_+^{-1} \cdot P_1, \dots, L_+^{-1} \cdot P_{s-2}$, for a certain “gauge transformation” $L_+ = Q_0(I + \hbar Q_1 + \dots)$. In contrast to the situation of section 4, there is no short cut to finding L_+ here. However, L_+ can be found as the unique solution of the system of ordinary differential equations

$$\frac{1}{\hbar} \hat{\omega} = L_+ \Omega L_+^{-1} + L_+ dL_+^{-1}$$

which satisfies the initial condition $L_+|_{q=0} = I$. It is proved in [14] that the system reduces to a system of algebraic equations for Q_0, Q_1, \dots which can be solved by an explicit algorithm.

Example 5.1. $X^3 \subseteq \mathbb{P}(1, 1, 1, 1, 1) = \mathbb{C}P^4$

In the notation of [14] this is M_5^3 . As this example is worked out in detail in Examples 3.6, 5.4, 6.24, 6.36 of [9] we shall just summarize the results of the calculations.

First, we have the differential operator

$$q^{-1} T_{1,1,1,1} - S_3 = q^{-1} (\hbar \partial)^4 - 3^3 \hbar^2 (\partial + \frac{1}{3}) (\partial + \frac{2}{3}).$$

With respect to the basis $P_0 = 1, P_1 = \hbar \partial, P_2 = (\hbar \partial)^2, P_3 = (\hbar \partial)^3$ the connection matrix is

$$\Omega = \frac{1}{\hbar} \begin{pmatrix} 1 & & 6q\hbar^2 \\ & 1 & 27q\hbar \\ & & 27q \\ & & & 1 \end{pmatrix}.$$

The gauge transformation $L_+ = Q_0(I + \hbar Q_1)$ can be found by solving the o.d.e. $\frac{1}{\hbar} \hat{\omega} = L_+ \Omega L_+^{-1} + L_+ dL_+^{-1}$ subject to $L_+|_{q=0} = I$. This gives

$$Q_0 = \begin{pmatrix} 1 & & 6q & \\ & 1 & 21q & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} & & 6q & \\ & & & \\ & & & \\ & & & \end{pmatrix}.$$

The new basis is $\hat{P}_0 = 1, \hat{P}_1 = \hbar\partial, \hat{P}_2 = (\hbar\partial)^2 - 6q, \hat{P}_3 = (\hbar\partial)^3 - 21q\hbar\partial - 6\hbar q$, and the matrix of the connection form is

$$\hat{\Omega} = \frac{1}{\hbar} \begin{pmatrix} & 6q & & 36q^2 \\ 1 & & 15q & \\ & 1 & & 6q \\ & & 1 & \end{pmatrix}$$

with respect to this basis. \square

The basis $\hat{P}_0, \dots, \hat{P}_{s-2}$ allows us to construct a product operation as in section 4. Unfortunately, this product does not in general satisfy the Frobenius property³ with respect to the natural pairing (which will be defined below). For this reason it is necessary to modify the basis further.

Step 2 For $w_1 = \dots = w_n = 1$ (see [14]), Step 1 is based on the Birkhoff factorization $L = L_- L_+$, where L is a map such that $L^{-1}dL = \Omega$, and Step 2 is unnecessary. For general w_1, \dots, w_n (but still with the assumptions of this section), the combined effects of Step 1 and Step 2 may be described in terms of the general Birkhoff factorization

$$L = L_- \gamma L_+$$

where $\gamma(\hbar) = \text{diag}(\hbar^{a_0}, \dots, \hbar^{a_{s-2}})$. Step 1 amounts to applying the gauge transformation L_+^{-1} to Ω , giving $\hat{\Omega} = \frac{1}{\hbar}\hat{\omega}$; Step 2 amounts to applying another gauge transformation G^{-1} to $\hat{\Omega}$, giving a connection matrix $\tilde{\Omega} = \frac{1}{\hbar}\tilde{\omega}$ which satisfies the required condition.

To explain G , we must review the Birkhoff decomposition

$$\Lambda GL_{s-1}\mathbb{C} = \bigcup_{\gamma \in \check{T}} \Lambda_- GL_{s-1}\mathbb{C} \gamma \Lambda_+ GL_{s-1}\mathbb{C}$$

(Theorem 8.1.2 of [13]), where \check{T} denotes the set of homomorphisms from S^1 to the diagonal matrices in $GL_{s-1}\mathbb{C}$. If γ is restricted to an appropriate subset, e.g. the homomorphisms satisfying $a_0 \leq \dots \leq a_{s-2}$, then this decomposition is a disjoint union. The “big cell” is the piece given by $\gamma = I$; it is a dense open subset of the identity component of $\Lambda GL_{s-1}\mathbb{C}$. The “small cells” (where $\gamma \neq I$) have finite codimension in $\Lambda GL_{s-1}\mathbb{C}$.

The term “cell” is used here because the decomposition is equivalent to the $\Lambda_- GL_{s-1}\mathbb{C}$ -orbit decomposition

$$\Lambda GL_{s-1}\mathbb{C} / \Lambda_+ GL_{s-1}\mathbb{C} = \bigcup_{\gamma \in \check{T}} \Lambda_- GL_{s-1}\mathbb{C} [\gamma]$$

of the infinite-dimensional Grassmannian $Gr^{(s-1)} \cong \Lambda GL_{s-1}\mathbb{C} / \Lambda_+ GL_{s-1}\mathbb{C}$ (see section 8.3 of [13]). It is analogous to the cell decomposition,

³More precisely, the natural pairing is not \mathbb{R} -valued on this basis.

or cell-bundle⁴ decomposition, of a finite-dimensional generalized flag manifold $G_{\mathbb{C}}/P$ given by the orbits of an opposite parabolic subgroup.

For (orbifold) quantum cohomology, a modification of $Gr^{(s-1)}$ is needed (see sections 5.3 and 10.4 of [9]), but we shall just explain the procedure in the case of $Gr^{(s-1)}$ itself, from which all other cases follow. The main point is that the “cell” $\Lambda_- GL_{s-1} \mathbb{C} [\gamma]$ is diffeomorphic to a unipotent subgroup Λ_-^γ of $\Lambda_- GL_{s-1} \mathbb{C}$ (Theorem 8.6.3 of [13]). This shows that any map L which takes values in $\Lambda_- GL_{s-1} \mathbb{C} \gamma \Lambda_+ GL_{s-1} \mathbb{C}$ (and therefore admits at least one factorization $L = L_- \gamma L_+$) has a *canonical* factorization

$$L = L_-^c \gamma L_+^c.$$

The factor L_-^c is the “most economical” choice for L_- . The same phenomenon occurs for finite-dimensional generalized flag manifolds. The simplest example is $\mathbb{C}P^n$: the i -dimensional cell \mathbb{C}^i can be described as an orbit of the group of upper triangular matrices in $GL_{n+1} \mathbb{C}$, but more economically as an orbit of a certain i -dimensional unipotent subgroup (see chapter 14, part III, of [7]).

Step 2 amounts to extracting the economical factor L_-^c from $L_- \gamma$. More precisely, by Theorem 8.6.3 of [13], we can write

$$\begin{aligned} L &= L_- \gamma L_+ \\ &= L_-^c L_-^f \gamma L_+ \end{aligned}$$

where L_-^f denotes the “superfluous factor”; this is a polynomial in \hbar^{-1} and satisfies $L_-^f \gamma = \gamma L_+^f$ where L_+^f is polynomial in \hbar . (The canonical factorization of L is then given by $L = L_-^c \gamma L_+^c$ with $L_+^c = L_+^f L_+$.) To find the desired map \tilde{L} , such that $\tilde{L}^{-1} d\tilde{L} = \tilde{\Omega} = \frac{1}{\hbar} \tilde{\omega}$, we must factorize L_-^c further as

$$L_-^c = \tilde{L} E$$

where E is independent of \hbar .

Thus, Step 2 involves applying the gauge transformation $G^{-1} = (EL_-^f \gamma)^{-1}$. As in Step 1, we seek $G^{-1} = \gamma^{-1}(Z_0 + \frac{1}{\hbar} Z_1 + \dots)$ such that $G \hat{\Omega} G^{-1} + G dG^{-1}$ is of the form $\frac{1}{\hbar} \tilde{\omega}$. The condition

$$\frac{1}{\hbar} \tilde{\omega} = G \hat{\Omega} G^{-1} + G dG^{-1}$$

may be regarded as a system of o.d.e. for Z_0, Z_1, \dots , which we can attempt to solve as in Step 1.

At this point, however, we have not specified the homomorphism γ . For the moment we just assume that there exists some γ and some G

⁴The cell decompositions here arise from Morse functions; the cell-bundle decompositions arise from Morse-Bott functions.

of the above form, so that we have

$$\tilde{\Omega} = \frac{1}{\hbar} \tilde{\omega}$$

with respect to the new basis $\tilde{P}_0 = G^{-1} \cdot \hat{P}_0, \dots, \tilde{P}_{s-2} = G^{-1} \cdot \hat{P}_{s-2}$. Subject to this assumption, we are now ready to construct an “abstract orbifold quantum cohomology ring”, which will be isomorphic to $\mathbb{C}[b, q^{1/(s-d)}]/(w^w b^{s-1} - d^d q b^{k-1})$.

1. The product

Let A be the vector space with basis denoted by the symbols

$$\begin{aligned} &1, p, \dots, p^{u_1-1}; \\ &\mathbf{1}_{f_2}, \mathbf{1}_{f_2}p, \dots, \mathbf{1}_{f_2}p^{u_2-1}; \\ &\dots \\ &\mathbf{1}_{f_k}, \mathbf{1}_{f_k}p, \dots, \mathbf{1}_{f_k}p^{u_k-1} \end{aligned}$$

We define QA to be $A \otimes \mathbb{C}[q^{1/(s-d)}]$ with the product structure specified by saying that the matrix of multiplication by p is $\tilde{\omega}$.

2. The grading

The differential operator $T_{w_1, \dots, w_n} - qS_{d-1}$ is homogeneous of weight $2s - 2$, if we assign weights as follows: $|\hbar| = 2$, $|\partial| = 0$, $|q| = 2s - 2d$. The differential operators $\tilde{P}_0, \dots, \tilde{P}_{s-1}$ are also homogeneous, and their weights are by definition the degrees of the corresponding abstract orbifold quantum cohomology classes listed above. Since $\tilde{\omega}$ is homogeneous, this makes QA a graded ring.

3. Self-adjointness and the pairing

We have the following analogue of Proposition 4.1:

Proposition 5.2.

- (1) δ_{n-1} is a cyclic element of $\bar{\mathcal{M}}^*$ (that is, $D^{\hbar} \odot \delta_{n-1} = \bar{\mathcal{M}}^*$).
- (2) $(T_{w_1, \dots, w_n} - qS_{d-1}) \odot \delta_{n-1} = 0$.
- (3) The map $\mathcal{M} \rightarrow \bar{\mathcal{M}}^*$, $[P] \mapsto [P \odot \delta_{n-1}]$ is an isomorphism of D^{\hbar} -modules.

Proof. We omit the proof, which is similar to that of Proposition 4.1, and which will be given in a more general context elsewhere. \square

The pairing obtained from the natural composition $\mathcal{M} \times \mathcal{M} \rightarrow \bar{\mathcal{M}}^* \times \mathcal{M} \rightarrow \mathcal{O}$ will be normalized in a different way from Definition 4.2, in order to take account of the degree of the hypersurface:

Definition 5.3. $\langle\langle P, Q \rangle\rangle = \frac{d}{w_1 \dots w_n} (P \odot \delta_{n-1})(Q)$.

We can now explain the choice of the homomorphism γ in Step 2 above. It arises from the fact that the isomorphism $\mathcal{M} \rightarrow \bar{\mathcal{M}}^*$ is not “homogeneous”, in general. We must choose a_0, \dots, a_{s-2} with the property that the elements

$$\hbar^{-a_0} \hat{P}_0 \odot \delta_{n-1}, \dots, \hbar^{-a_{s-2}} \hat{P}_{s-2} \odot \delta_{n-1}$$

of $\bar{\mathcal{M}}^*$ have *minus* the weighted degrees of the elements $\hat{P}_0, \dots, \hat{P}_{s-2}$ (not necessarily in the same order). It is natural to impose the additional condition that a_j is independent of j for $u_1 + \dots + u_i \leq a_j \leq u_1 + \dots + u_{i+1} - 1$; in this case there is a unique solution

$$a_j = \frac{1}{2}(u_1 + \sum_{l=2}^i u_l + u_{i+1}) - (s-d)f_{i+1}$$

for $u_1 + \dots + u_i \leq a_j \leq u_1 + \dots + u_{i+1} - 1$. At this point we must extend the ring D^{\hbar} to $D^{\hbar, \hbar^{-1}}$ i.e. we allow negative powers of \hbar .

Example 5.4. $X^3 \subseteq \mathbb{P}(1, 1, 1, 2)$

We have $w_0 = w_1 = w_2 = 1$, $w_3 = 2$ and $s = 5$, $d = 3$. The differential operator is

$$q^{-1}T_{1,1,2} - S_3 = q^{-1}\hbar^4\partial^3(\partial - \frac{1}{2}) - 3^3\hbar^2(\partial + \frac{1}{3})(\partial + \frac{2}{3}).$$

We have $F = \{\frac{0}{1}, \frac{0}{1}, \frac{0}{2}, \frac{1}{2}\} = \{0, \frac{1}{2}\}$, so $u_1 = 3, u_2 = 1$. As in section 4 we can display the data as follows:

	$w_1 = 1$	$w_2 = 1$	$w_3 = 2$	
$S_{f_1} = \{1, 2, 3\}, f_1 = 0$	$\frac{0}{1}$	$\frac{0}{1}$	$\frac{0}{2}$	$\Delta_1 = \frac{1}{2}, m_1 = 2$
$S_{f_2} = \{3\}, f_2 = \frac{1}{2}$			$\frac{1}{2}$	$\Delta_2 = \frac{1}{2}, m_2 = 2$

The factorization of $q^{-1}T_{1,1,2}$ is

$$q^{-1}T_{1,1,2} = 2q^{-\frac{1}{2}}(\hbar\partial)2q^{-\frac{1}{2}}(\hbar\partial)^3 = \frac{1}{r}(\hbar\partial)\frac{1}{r}(\hbar\partial)^3,$$

where $r = \frac{1}{2}q^{\frac{1}{2}}$. Thus, our starting point is the basis $P_0 = 1$, $P_1 = \hbar\partial$, $P_2 = (\hbar\partial)^2$, $P_3 = \frac{1}{r}(\hbar\partial)^3$. We have $|r| = 2$, so the degrees of these basis elements are 0, 2, 4, 4. With respect to this basis we have

$$\Omega = \frac{1}{\hbar} \begin{pmatrix} 1 & 6\hbar^2 r \\ & 1 & 27\hbar r \\ & & r \end{pmatrix}.$$

Step 1 The gauge transformation L_+^{-1} is given by

$$L_+ = \begin{pmatrix} 1 & 12r^2 & & \\ & 1 & 30r & \\ & & 1 & \\ & & & 1 \end{pmatrix} \left(I + \hbar \begin{pmatrix} & 12r \\ & & \\ & & & \end{pmatrix} \right).$$

Application of L_+^{-1} produces the new basis $\hat{P}_0 = P_0$, $\hat{P}_1 = P_1$, $\hat{P}_2 = P_2 - 12r^2 P_0$, $\hat{P}_3 = P_3 - 30r P_1 - 12\hbar r P_0$. With respect to this basis, we have

$$\hat{\Omega} = \frac{1}{\hbar} \hat{\omega} = \frac{1}{\hbar} \begin{pmatrix} 12r^2 & -36r^3 \\ 1 & 18r^2 & -3r \\ & r & \end{pmatrix}.$$

We omit the details of this calculation, which is similar to those in [14].

Step 2 Since the degrees of $\hat{P}_0, \hat{P}_1, \hat{P}_2, \hat{P}_3$ are 0, 2, 4, 4, the degrees of $\hat{P}_0 \odot \delta_2, \hat{P}_1 \odot \delta_2, \hat{P}_2 \odot \delta_2, \hat{P}_3 \odot \delta_2$ are $-4, -2, 0, 0$, which are not the same as $-0, -2, -4, -4$. To remedy this, it suffices to replace \hat{P}_3 by $\hbar^{-1} \hat{P}_3$ and repeat the calculation. Thus, we take the homomorphism $\gamma(\hbar) = (1, 1, 1, \hbar)$.

To find the gauge transformation G^{-1} , which is necessarily of the form $G^{-1} = \gamma^{-1} Z = \gamma^{-1}(Z_0 + \frac{1}{\hbar} Z_1)$, we must attempt to solve the o.d.e.

$$\frac{1}{\hbar} Z \tilde{\omega} = \gamma \hat{\Omega} \gamma^{-1} Z + dZ$$

subject to the initial condition $Z|_{q=0} = \text{diag}(1, 1, 1, \frac{2}{3})$.

Equating the coefficients of each power of \hbar gives a collection of equations for the coefficients of Z_0, Z_1 and $\tilde{\omega}$. These equations can be solved, and produce

$$Z = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -2r & \frac{2}{3} \end{pmatrix} + \frac{1}{\hbar} \begin{pmatrix} & -6r^2 & 2r \\ & & & \end{pmatrix},$$

and we obtain

$$\tilde{\Omega} = \frac{1}{\hbar} \tilde{\omega} = \frac{1}{\hbar} \begin{pmatrix} 12r^2 & & \\ 1 & 12r^2 & r \\ & 1 & \\ & 3r & \end{pmatrix}.$$

This is the connection matrix with respect to the new basis $\tilde{P}_0 = 1, \tilde{P}_1 = \hbar \partial, \tilde{P}_2 = \hat{P}_2 - \frac{2r}{\hbar} \hat{P}_3 - \frac{6r^2}{\hbar} \hat{P}_1, \tilde{P}_3 = \frac{2}{3\hbar} \hat{P}_3 + \frac{2r}{\hbar} \hat{P}_1$ (obtained by applying G^{-1} to the previous basis $\hat{P}_0, \hat{P}_1, \hat{P}_2, \hat{P}_3$).

Explicit expressions for L_-^f, E can be read off from the factorization $G^{-1} = (EL_-^f \gamma)^{-1} = \gamma^{-1} (L_-^f)^{-1} E^{-1}$, which is

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \frac{1}{\hbar} \end{pmatrix} \begin{pmatrix} 1 & -\frac{3}{2}r^2 & -\frac{3}{2}r \\ & 1 & \frac{1}{\hbar} 3r \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{3}{2}r^2 & r \\ & 1 & \\ & & 1 \\ & -2r & \frac{2}{3} \end{pmatrix}.$$

Now we can construct our product operation on $A \otimes \mathbb{C}[r]$, where A is the vector space with basis elements $1, p, p^2, \mathbf{1}_{\frac{1}{2}}$. The matrix of structure constants (abstract orbifold quantum multiplication by p) with respect to this basis is, by definition, the matrix $\tilde{\omega}$.

These products determine all others, and we obtain the following orbifold quantum multiplication table:

	1	p	p^2	$\mathbf{1}_{\frac{1}{2}}$
1	1	p	p^2	$\mathbf{1}_{\frac{1}{2}}$
p		$p^2 + 12r^2 + 3r\mathbf{1}_{\frac{1}{2}}$	$12r^2p$	rp
p^2			$108r^4 + 36r^3\mathbf{1}_{\frac{1}{2}}$	$12r^3$
$\mathbf{1}_{\frac{1}{2}}$				$\frac{1}{3}p^2 - 3r\mathbf{1}_{\frac{1}{2}}$

Defining $\text{age } \mathbf{1}_{f_i} = \frac{1}{2}|\tilde{P}_{u_1+\dots u_{i-1}}|$, we obtain:

$\text{age } \mathbf{1}_0 = 0$	$ 1 = 0$	$ p = 2$	$ p^2 = 4$
$\text{age } \mathbf{1}_{\frac{1}{2}} = 1$			$ \mathbf{1}_{\frac{1}{2}} = 2$

These results agree with the orbifold Gromov-Witten computations of Corti ([4]).

It follows, in particular, that the Frobenius property must hold for our product operation. We conclude by indicating why this follows independently from our D-module calculation. First, by direct calculation from Definition 5.3, we obtain

$$\left(\langle \hat{P}_i, \hat{P}_j \rangle \right)_{0 \leq \alpha, \beta \leq 3} = \begin{pmatrix} & & \frac{3}{2} & \\ & \frac{3}{2} & & -\frac{9}{2}r \\ \frac{3}{2} & & \frac{9}{2}r^2 & \frac{9}{4}\hbar r \\ & -\frac{9}{2}r & -\frac{9}{2}\hbar r & -\frac{9}{8}\hbar^2 + \frac{27}{2}r^2 \end{pmatrix}$$

and

$$\left(\langle \tilde{P}_i, \tilde{P}_j \rangle \right)_{0 \leq \alpha, \beta \leq 3} = \begin{pmatrix} & & \frac{3}{2} & \\ & \frac{3}{2} & & \\ \frac{3}{2} & & & \\ & & & \frac{1}{2} \end{pmatrix} \quad (= S, \text{ say}).$$

By construction (see the discussion following Definition 6.14 of [9]), our product satisfies the Frobenius property with respect to the pairing whose matrix is S . The latter happens to agree with the matrix of the orbifold Poincaré pairing from [4].

This gives rise to an alternative interpretation of Step 2 of our procedure. Namely, starting from $\hat{P}_0, \hat{P}_1, \hat{P}_2, \hat{P}_3$ we apply the Gram-Schmidt orthonormalization procedure to obtain a basis O_0, O_1, O_2, O_3 which is “orthonormal” with respect to the above pairing in the sense that $\langle\langle O_i, O_j \rangle\rangle = \frac{3}{2}$ for $i + j = 2$ and $\langle\langle O_3, O_3 \rangle\rangle = \frac{1}{2}$ (and $\langle\langle O_i, O_j \rangle\rangle = 0$ otherwise). There are many ways to do this, but a further Birkhoff factorization (of the form $L = L_- L_+$) always produces the above basis $\tilde{P}_0, \tilde{P}_1, \tilde{P}_2, \tilde{P}_3$.

The explanation for the uniqueness of $\tilde{P}_0, \tilde{P}_1, \tilde{P}_2, \tilde{P}_3$ is as follows. Our original version of Step 2 involves a Birkhoff factorization of the form $L = L_- \gamma L_+$. The Frobenius property is satisfied if and only if L_- is a twisted loop with respect to the involution defined by S , i.e. $S^{-1}(L_-^t)^{-1}S = L_-(-\hbar)$ (section 6.5 of [9]). Now, if there exists *some* twisted L , for example, from any Gram-Schmidt orthonormalization, and γ is twisted, then L_- must also be twisted, as the Birkhoff decomposition is valid also for the twisted loop group. By the uniqueness of the (canonical form of the) Birkhoff decomposition $L = L_-^c \gamma L_+^c$, we always obtain the same L_- . \square

REFERENCES

- [1] A. Amarzaya and M. A. Guest, *Gromov-Witten invariants of flag manifolds, via D-modules*, J. London Math. Soc. **72** (2005), 121–136 (math.DG/0306372).
- [2] S. Boissière, È. Mann, and F. Perroni, *On the cohomological crepant resolution conjecture for weighted projective spaces*, preprint, math.AG/0610617.
- [3] T. Coates, A. Corti, Y.-P. Lee, and H.-H. Tseng, *The quantum orbifold cohomology of weighted projective space*, preprint, math.AG/0608481.
- [4] A. Corti, Lecture at UK-Japan Winter School, Warwick University, January 2008.
- [5] A. Corti and V. Golyshev, *Hypergeometric equations and weighted projective spaces*, preprint, math.AG/0607016.
- [6] V. Golyshev, *Classification problems and mirror duality*, preprint, math.AG/0510287.
- [7] M. A. Guest, *Harmonic Maps, Loop Groups, and Integrable Systems*, LMS Student Texts 38, Cambridge Univ. Press, 1997.
- [8] M. A. Guest, *Quantum cohomology via D-modules*, Topology **44** (2005) 263–281 (math.DG/0206212).
- [9] M. A. Guest, *From Quantum Cohomology to Integrable Systems*, Oxford Graduate Texts in Math. 15, Oxford Univ. Press, 2008.
- [10] H. Iritani, *Quantum D-modules and equivariant Floer theory for free loop spaces*, Math. Z. **252** (2006), 577–622 (math.DG/0410487).
- [11] T. Kawasaki, *Cohomology of twisted projective spaces and lens complexes*, Math. Ann. **206** (1973), 243–248.

- [12] È. Mann, *Orbifold quantum cohomology of weighted projective spaces*, J. Algebr. Geom. **17** (2008), 137–166 (math.AG/0610965).
- [13] A. N. Pressley and G. B. Segal, *Loop Groups*, Oxford Univ. Press, 1986.
- [14] H. Sakai, *Gromov-Witten invariants of Fano hypersurfaces, revisited*, J. Geom. Phys. **58** (2008), 654–669 (math.DG/0602324).

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